Acausal relations between topographic slope and drainage area

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Abstract. For real landscapes, the local slope is related to the area of the drainage basin. Such relations do not always indicate a causal link between drainage area and local slope. We provide a general and quantitative assessment of this statistical effect. Slope-area relations in which slopes decay slower than roughly the cube root of drainage area are insensitive to processes that have shaped the landscape. To illustrate this point, surfaces unrelated to erosion are compared to real landscapes. The fluctuations of slope for constant drainage area provide a means of distinguishing between causal and random effects.

1. Introduction

Although the dendritic structure of river basins is of great natural beauty, quantitative measures of network structure (*Rodriguez-Iturbe and Rinaldo* [1997]) have proven notoriously difficult to link to unambiguous physical mechanisms (*Kirchner* [1993]; *Dodds and Rothman* [2000]). On the other hand, much physical significance has been associated with a reported power-law relation $S \sim A^{-\theta}$ between slope S and drainage area A. Because fluvial erosion by surface runoff shapes landscapes, it can create a systematic relation between local slope and the discharge of water. Mechanistic models of river incision on steady-state topography lead to specific values of θ (e.g., *Flint* [1974]; *Willgoose et al.* [1991]; *Howard et al.* [1994]; *Banavar et al.* [1997]). However, if streams merely aggregate, without much alteration of the landscape topography, there is still a systematic relation between slope and area (*Schorghofer and Rothman* [2001]). This is due to the aggregative geometry of downhill flow. In *Schorghofer and Rothman* [2001] we have given a first qualitative description of this phenomenon. In the present work, the basic effect is rederived in a way that is more general yet simpler, quantitative results for the exponent θ are presented, and a specific procedure to test the significance of any causal connection is outlined.

2. The slope-curvature relation

The curvature κ of contour lines measures the "focusing strength" or convergence of the surface topography (*Wilson and Gallant* [2000]; *Schorghofer and Rothman* [2001]). Hence, positive and negative curvature correspond, respectively, to convergent and divergent topography and higher curvature corresponds to stronger convergence. If the elevation is denoted by $z(x, y)$, then the curvature κ of contour lines (i.e., the plan curvature) may be written as (*Wilson and Gallant* [2000]; *Bronshtein and Semendyayev* [1985])

$$\kappa = \frac{(\partial_y h)^2 \partial_{xx} h - 2(\partial_x h)(\partial_y h)\partial_{xy} h + (\partial_x h)^2 \partial_{yy} h}{\left[(\partial_x h)^2 + (\partial_y h)^2\right]^{3/2}},$$

or equivalently,

$$\kappa = \frac{(\nabla z)^t \begin{pmatrix} \partial_{yy} z & -\partial_{xy} z \\ -\partial_{xy} z & \partial_{xx} z \end{pmatrix} (\nabla z)}{|\nabla z|^3}. \quad (1)$$

Here t denotes the transpose and $|\nabla z| = S$. Equation (1) provides insight into the slope dependence of the convergence. There are two factors of slope in the numerator, but three powers of slope in the denominator. Therefore, one expects a correlation between κ and $|\nabla z|$.

As a comprehensive example we consider all possible surfaces on which first and second derivatives are statistically independent of one another. Indeed such behavior is very typical and the most important example of surfaces with this property are Gaussian surfaces (*Longuet-Higgins* [1957]; *Adler* [1981]), defined in section 4. Since the derivatives are statistically independent, the second derivatives in eq. (1) reduce to a prefactor and it follows directly that $\kappa^2 \sim 1/S^2$, or approximately,

$$|\kappa| \sim 1/S. \quad (2)$$

Hence the naive expectation that κ depends on S is exactly realized.

Relation (2) is often valid for real landscapes (*Schorghofer and Rothman* [2001]). For example, figure 1 shows the dependence of curvature and second derivatives on slope for a real drainage basin. As for our example surfaces, the curvature depends strongly on the slope, while second derivatives show a much weaker dependence. It is therefore a far better approximation to assume the derivatives are independent than it would be to think of curvature and slope as uncorrelated.

The general underlying idea is that the expression for the curvature (1) contains the slope. Such a dependence could only be avoided if a particular correlation exists between the first and second derivative terms. Otherwise one must have a strong positive correlation between κ and $1/S$. The strong correlation between κ and S is therefore a basic geometric tendency.

According to eq. (2), flat slopes have, on average, high curvature and steep slopes have, on average, low curvature. This is the opposite of what one may intuitively expect. Note, however that a sheet of paper held at shallow slopes

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has a high plan curvature even if only slightly bent. Figure 2 illustrates the effect. Intuitively, water is easily deflected on a small slope, while it undergoes little sideways focusing on steep slopes. Equation (2) shows that the slope-curvature effect demonstrated in figure 2 is typical of many surfaces.

3. A non-causal slope-area relation

The curvature is a measure of the local accumulation of the topography, while the drainage area is literally its global accumulation. It is hence suggestive that curvature κ and contributing area A are positively correlated. Quantitatively, the curvature κ is a measure of how much contour lines shrink per unit horizontal spacing (*Schorghofer and Rothman* [2001]). A concentration of flow leads to more flow and flow (discharge) is proportional to the contributing area. Figure 3 illustrates this connection.

Figure 4 shows slope-area relations for a Gaussian surface, a fracture surface (*Lopez and Schmittbuhl* [1998]), and the Juan River basin. In the first two examples the flow is stopped at local minima, while the last example consists of a single basin only. The fracture is created by driving a crack through a granite sample. On such a surface there definitely cannot be any causal relation between drainage area and slope, because their is no flow of water on the surface. Yet, the slope-area relation is about equally pronounced in all three examples, for all but the largest contributing areas in the Juan River. This is a compelling illustration of the independence of a slope-area relation from specific physical processes.

Incidentally, all graphs in figure 4 have an initial upward trend. Equation (2) implies not only that more convergent regions ($\kappa > 0$) are associated with smaller slopes, but also that more *divergent* regions ($\kappa < 0$) are associated with smaller slopes. Hence, extremely small contributing areas are associated, on average, with extremely small slopes. This is the reason for the initial growth seen in figure 4. The upward trend stops soon because divergent regions do not accumulate much water.

Landscape morphology can be the result of different processes. On any topography, curvature and slope help to define stream locations, independent of how the geometry was created. Hence, one expects the presence of a slope-area relation that is largely process-independent. Many different processes that shape landscapes lead to observation of the same slope-area relation.

For this geometric tendency there is no causal connection between contributing area and slope. Rather than the area determining the slope, the slope systematically collects from areas according to their size. In a sense, streams form preferentially along paths of smaller slope. This "acausal" or "statistical" slope-area effect has no relation to causal or deterministic slope-area theories (e.g. *Flint* [1974]; *Willgoose et al.* [1991]; *Howard et al.* [1994]; *Banavar et al.* [1997]).

A fundamental difference between deterministic and statistical slope-area relations is the nature of fluctuations. In the former case, fluctuations arise due to variations in geological parameters. If the slope-area effect is deterministic it must be possible to explain variations in the slope by variations in geological parameters. In the latter case, the fluctuations are intrinsically statistical. Rivers "happen" to end up on different slopes. The presence of large fluctuations in the slope-area relation of real landscapes may be the most direct signature of non-causal effects.

4. Quantitative Null Hypothesis

The above considerations are mostly qualitative. For quantitative results we consider Gaussian surfaces. A surface is called Gaussian if the phases of its Fourier modes are random and uniformly distributed (*Longuet-Higgins* [1957]; *Adler* [1981]). The amplitudes of the modes are however arbitrary. Since the amplitudes fully determine the spatial correlations and the power spectrum, there are Gaussian surfaces for every realistic spatial correlation.

The theoretical motivation for choosing a power-law, $S \sim A^{-\theta}$, for the dependence of S on A is the approximate scale-invariance of landscapes. Consequently, the power spectrum of the landscape should also be assumed to be a power law. Such spectra can be specified by a Hurst exponent H (*Turcotte* [1997]), which for a real landscape can be determined from the power spectrum or, for $H < 1$, from the spatial correlation function, $\left\langle (h(r+R) - h(r))^2 \right\rangle \sim R^{2H}$, where the average is taken over the spatial domain. Figure 5 provides quantitative results for the exponent θ as a function of the Hurst exponent H. From this figure it is clear that neither the real landscape nor the fracture surface deviate significantly from the Gaussian behavior. In fact, θ for the Juan River basin is no different from that of a fracture surface.

In a quantitative comparison one has to distinguish between averages taken over the entire basin and averages taken only along a single stream. The two averages coincide for large but not for small drainage areas. Our theory is only applicable to basin-averaged quantities. Figure 4 shows a transition in the slope-area relation for the Juan River basin (circles). If the power-law $S \sim A^{-\theta}$ is abandoned in favor of a local definition $\theta = -d \log S / d \log A$, where θ is allowed to be a function of A, then θ varies in the range of $0.11 - 0.33$ for the Juan River basin. (Depending on how the slopes are determined, slightly higher values for θ can be found). The value of θ plotted in figure 5 for the Juan River basin corresponds to a fit over almost the entire range of available areas. For terrestrial incised channels, values of θ are typically in the range of $0.35 - 0.7$ (*Snyder et al.* [2000]), but terrains away from major incised channels often have lower values of θ, as seen in the example of the Juan River. We have studied a number of basins in Northern California. The slope-area dependence on much of these terrains is not significantly different from that on random (Gaussian) landscapes. Thus, if Gaussian landscapes are viewed as a null hypothesis for the structure of these terrains, we find no evidence to reject it. However, if we consider only the highest contributing areas (i.e., the incised channels) this acausal model is insufficient, indicating that other, presumably physical, mechanisms have shaped the channels.

Stated more generally, the Gaussian null hypothesis may be rejected if measurements of θ depart significantly from the line in figure 5. More simply, observations of $\theta \lesssim 1/3$ should not be considered indicative of specific erosion mechanisms in the absence of other information. An example is the slope-area data of *Montgomery and Dietrich* [1989] shown in figure 6. Slopes and areas are measured just above channel heads. The behavior in figure 6 is consistent with a mere acausal relation between slope and contributing area.

The situation is perhaps analogous to the fractal dimension of landscapes. A wide variety of surfaces are observed to be fractal, including those related and unrelated to erosion (*Turcotte* [1997]; *Dodds and Rothman* [2000]). Much of the same appears to be true for observations of $\theta \lesssim 1/3$.

Fluctuations may also be analyzed quantitatively on Gaussian surfaces. For Gaussian surfaces the derivatives are distributed as Gaussians and, as already noted, are independent of each other (*Longuet-Higgins* [1957]; *Adler* [1981]). If σ is the standard deviation for the distribution of $\partial z/\partial x$ and $\partial z/\partial y$, then the probability distribution of the slope is $(S/\sigma^2)\exp(-S^2/2\sigma^2)$. The fluctuations relative to the average slope are consequently $\left\langle (S-\langle S\rangle)^2\right\rangle / \langle S\rangle^2 = 4/\pi - 1 \approx 0.52^2$. Relative to the median slope they are $\left\langle (S-S_M)^2\right\rangle / S_M^2 = 1/\ln 2 - 1 \approx 0.67^2$. Hence the fluctuations are about half as big as the average slope and about two-thirds that of the median slope. Empirically, one usually finds that the variance of slopes decreases with area slower than the (square of) the average slope. Therefore, the relative fluctuations increase with area, but the above mean quantities can serve as a rule of thumb.

5. Conclusions

When the slope-area relation is weak, it is insensitive to the processes that have shaped the terrain. The presence of a slope-area dependence can even be entirely unrelated to erosive processes. Most illustrative, some terrestrial basins have slope-area relations that are indistinguishable from that of a fracture surface. A departure from the line in figure 5 is necessary for any inferences of physical processes that have shaped landscapes. This result is directly applicable not only to the analysis of terrestrial erosive processes, but also to the current debate concerning the processes that have shaped landscapes on Mars (*Baker* [2001]; *Aharonson et al.* [2002]).

The acausal slope-area relation detailed above is not strong enough to quantitatively account for the pronounced slope-area relations found for many incised channels on Earth. However, it provides a rationale for observed slope-area relations when the exponent θ is small, and, in this way, supplements mechanistic approaches to river incision and basin formation.

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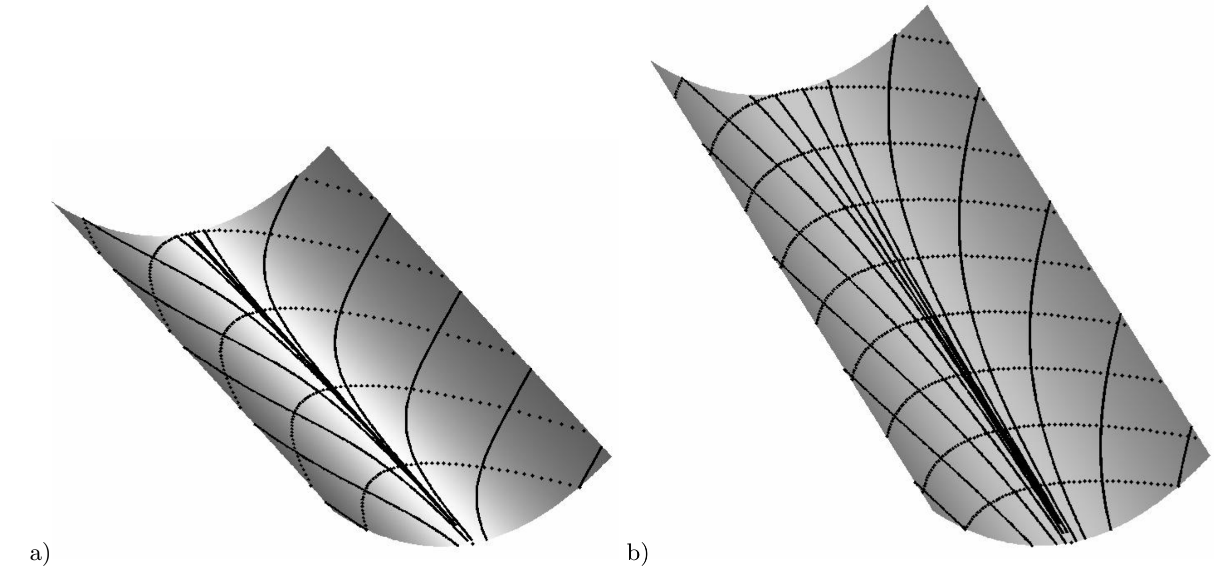

Figure 2. Parabolic surfaces with height contours (dotted lines) and streamlines (solid lines). The surface in (a) is less steeply sloped than the surface in (b) and therefore collects water into the main stream more rapidly. The shading of the surface reflects the plan curvature. Bright areas indicate a high curvature of contours; dark areas indicate low plan curvature.

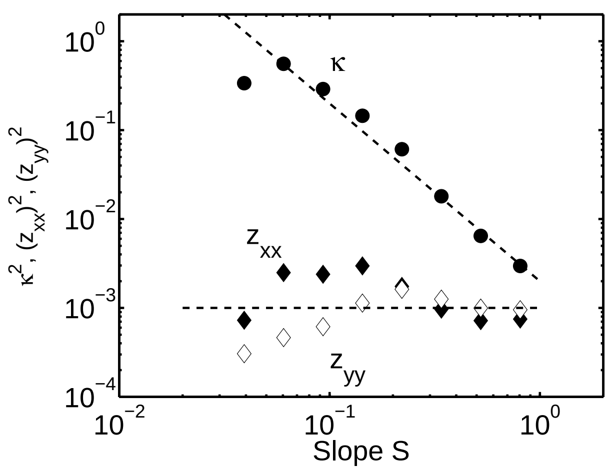

Figure 1. Dependence of curvature (circles) and second derivatives (diamonds) on slope for the Juan River basin, California. The straight lines indicate the Gaussian behavior. The data derive from a digital elevation map with 30m resolution.

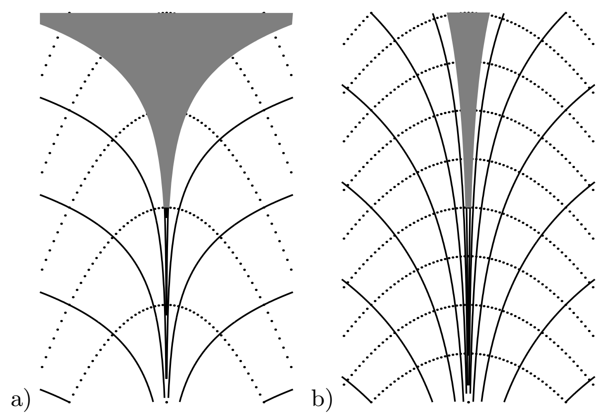

Figure 3. Plan view of the surfaces shown in figure 2. The shaded area is the contributing area of a patch which has the same size in (a) and (b). This illustrates the connection between slope, plan curvature, and contributing area.

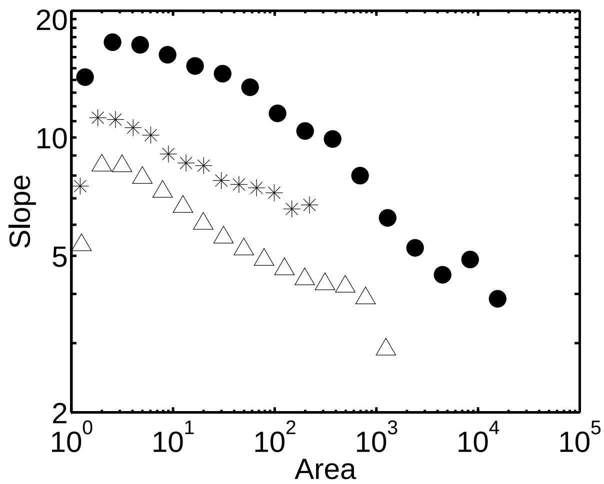

Figure 4. Triangles show the slope-area relation on a Gaussian surface. The stars indicate a fracture surface (*Lopez and Schmittbuhl* [1998]). Circles correspond to the Juan River basin, California. The area is in units of grid cells; slopes are multiplied by an arbitrary prefactor.

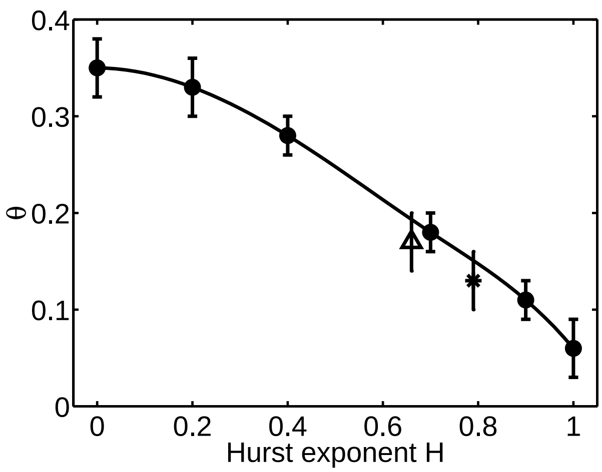

Figure 5. Circles show the slope-area effect on Gaussian surfaces over the range of Hurst exponents of natural landscapes. If no such effect were present, the exponent θ would be zero. The triangle with error bars corresponds to the Juan River basin, California. The star indicates a fracture surface (*Lopez and Schmittbuhl* [1998]). Note that, with the exception of the real landscape, none of these surfaces is shaped by erosion.

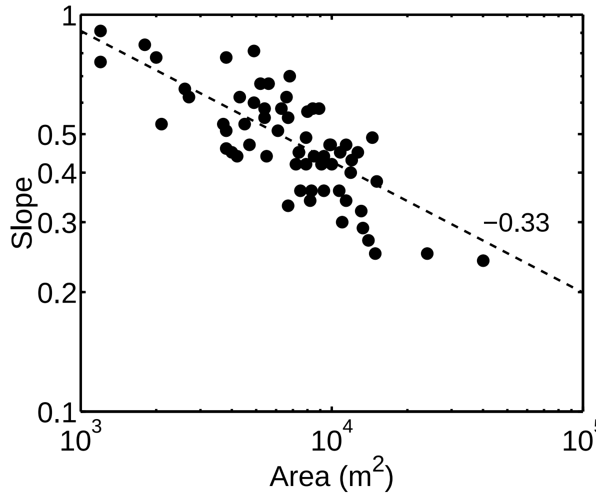

Figure 6. Slope-area data of the Tennessee Valley, California according to *Montgomery and Dietrich* [1989]. Data are acquired just above channel heads. The dashed line is the least-square fit of a power-law to the data.